Minimal unitary models and the closed $SU(2)_q$ invariant spin chain

1996 J. Phys. A: Math. Gen. 29 1187

(http://iopscience.iop.org/0305-4470/29/6/007)

J. Phys. A: Math. Gen. **29** (1996) 1187–1199. Printed in the UK

Minimal unitary models and the closed $SU(2)_q$ invariant spin chain

Silvio Pallua† and Predrag Prester‡

Department of Theoretical Physics, University of Zagreb, POB 162, Bijeniçka c.32, 41001 Zagreb, Croatia

Received 9 May 1995, in final form 20 November 1995

Abstract. We consider the Hamiltonian of the closed $SU(2)_q$ invariant chain. We project a particular class of statistical models belonging to the unitary minimal series. A particular model corresponds to a particular value of the coupling constant. The operator content is derived. This class of models has charge-dependent boundary conditions. In simple cases (Ising, three-state Potts) corresponding Hamiltonians are constructed. These are non-local as the original spin chain.

1. Introduction

Quantum groups together with the Temperly–Lieb algebra play a particular role in integrable spin chains [1]. However, it may be interesting to study particular Hamiltonians which are invariant to the quantum group [1–5]. The quantum group invariant Hamiltonian for the closed spin chain was constructed by Martin and Rittenberg [6]. This model was independently investigated in [7, 8]. It was shown that the properties of the ground state were such that for special values of the coupling constant, conformal anomalies of minimal unitary theories were obtained. In addition, this Hamiltonian implied boundary conditions which depended on the coupling constant (or quantum group parameter q) and quantum numbers of the sector. This second property made this Hamiltonian different from the XXZ chain with the toroidal boundary condition where the twist was common to all sectors of a given Hamiltonian [9–12]. In this paper we proceed with this investigation and show that it is possible to project from the closed quantum chain partition functions of statistical models corresponding to minimal unitary theories. In the finite-size scaling limit, we obtain the spectra and the operator content of these theories. For finite chains, the spectra of these models can be related to the starting quantum chain. Like the original XXZ chain, the projected systems also have sector-dependent boundary conditions. In our derivation we try to exploit the theory of representations of quantum groups [1, 14, 15] and the division of all states into 'good' and 'bad'. Keeping only 'good' states will lead to unitary theories.

† E-mail: pallua@phy.hr
‡ E-mail: pprester@phy.hr

0305-4470/96/061187+13$19.50

2. Statistical systems and the quantum chain

We start with the Hamiltonian for the closed $SU(2)_q$ invariant chain [5, 7]

$$H = Lq - \sum_{i=1}^{L-1} R_i - R_0 \tag{1}$$

$$R_0 = G R_{L-1} G^{-1} \tag{2}$$

$$G = R_1 \cdots R_{L-1} \tag{3}$$

where R_i are 4×4 matrices

$$R_i = \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \frac{q+q^{-1}}{4}(\sigma_i^3 \sigma_{i+1}^3 + 1) - \frac{q-q^{-1}}{4}(\sigma_i^3 - \sigma_{i+1}^3 - 2). \tag{4}$$

We choose the quantum group parameter q to be on the unit circle

$$q = \mathrm{e}^{\mathrm{i}\varphi} \qquad \frac{q+q^{-1}}{2} = \cos\varphi = -\cos\gamma. \tag{5}$$

The Hamiltonian is invariant to generators of the quantum group

$$S^3 = \tfrac{1}{2} \sum_{i=1}^{L} \sigma_i^3 \qquad S^{\pm} = \sum_{i=1}^{L} q^{-\sigma^3/2} \otimes \cdots \otimes q^{-\sigma^3/2} \otimes \sigma_i^{\pm} \otimes q^{\sigma^3/2} \otimes \cdots \otimes q^{\sigma^3/2}. \tag{6}$$

The operator G plays the role of the translation operator

$$G R_i G^{-1} = R_{i+1} \qquad R_L = R_0 \qquad i = 1, \ldots, L-1 \tag{7}$$

and also commutes with the Hamiltonian. We shall be interested in the cases in which the quantum group parameter is a root of unity:

$$q^n = \pm 1. \tag{8}$$

We shall first study the generic irrational case. In this case, one can decompose the space of states into the direct sum of irreducible representations of the quantum group which are in one-to-one correspondence with the usual $SU(2)$ representations. It is therefore sufficient to treat the highest weight states. All other states can be obtained with the action of the S^- operator. We derived the Bethe ansatz (BA) equation in [7]. In this reference the energy eigenvalues are given by

$$E = 2 \sum_{i=1}^{M} (\cos\varphi - \cos k_i) \qquad M = \frac{L}{2} - Q. \tag{9}$$

Here Q is the eigenvalue of S^3 and k_i satisfy the BA constraints

$$L k_i = 2\pi I_i + 2\varphi(Q+1) - \sum_{\substack{j=1 \\ j \neq i}}^{M} \Theta(k_i, k_j) \qquad k_i \neq \varphi \tag{10}$$

where I_i are integers (half-integers) if M is odd (even), and $\Theta(k_i, k_j)$ is the usual two-particle phase defined in [7].

It is important to notice that the BA functions $\Psi_M(n_1, \ldots, n_M)$ satisfy non-trivial boundary conditions:

$$\Psi_M(n_2, \ldots, n_M, n_1 + L) = \mathrm{e}^{\mathrm{i}\phi}\Psi_M(n_1, \ldots, n_M) \tag{11}$$

where the numbers n_i denote the positions of down spins and

$$\phi = 2\varphi(Q + 1). \tag{12}$$

This means that quantum invariance implies a non-trivial boundary condition. This boundary condition has two properties. It depends on the coupling constant

$$\gamma = \pi - \varphi \tag{13}$$

and on the sector defined by the charge Q.

Owing to the antisymmetry of phase shifts, from (10) it follows that

$$\sum_{i=1}^{M} k_i = \frac{2\pi}{L}\sum_{i=1}^{M} I_i + \frac{2M}{L}\varphi(Q + 1). \tag{14}$$

This allows us to determine the eigenvalues of the translation operator G or equivalently of the operator P

$$P = \mathrm{i}\ln G. \tag{15}$$

In fact,

$$\begin{aligned} P &= \sum_{i=1}^{M} k_i - \varphi\left(Q - 1 + \frac{L}{2}\right) \\ &= \frac{2\pi}{L}\sum_{i=1}^{M} I_i + \varphi\left[-\frac{L}{2} - Q + 1 + \frac{2M}{L}(Q + 1)\right]. \end{aligned} \tag{16}$$

It was also shown in [7] that the finite-size correction to the thermodynamic limit of the ground-state energy was given by (L even)

$$E_0(L) = E_0(\infty) - \frac{\pi c\,\zeta}{6L} + \mathrm{O}\left(\frac{1}{L}\right) \tag{17}$$

where

$$\zeta = \frac{\pi\sin\gamma}{\gamma}. \tag{18}$$

The conformal anomaly c was found to be

$$c = 1 - \frac{6(\pi - \varphi)^2}{\pi\varphi} \tag{19}$$

for $\varphi \in [\frac{\pi}{2}, \pi]$. We are particularly interested in the values

$$\varphi = \frac{\pi m}{m + 1} \qquad m = 3, 4, \ldots \tag{20}$$

because they give the conformal anomalies of the minimal unitary models:

$$c = 1 - \frac{6}{m(m + 1)} \qquad m = 3, 4, \ldots. \tag{21}$$

Now we define scaled gaps

$$\overline{E}_n = \frac{L}{2\pi\zeta}(E_n - E_0) \tag{22}$$

$$\overline{P}_n = \frac{L}{2\pi}(P_n - P_0 + \varphi Q). \tag{23}$$

We introduce the partition function in some sector $Q \geqslant 0$:

$$\mathcal{F}_Q(z,\bar{z},L) = \sum_{\text{all states}} z^{\frac{1}{2}(\overline{E}_n+\overline{P}_n)} \bar{z}^{\frac{1}{2}(\overline{E}_n-\overline{P}_n)}. \tag{24}$$

One can also introduce the partition function $\mathcal{K}_Q(z,\bar{z},L)$ for the highest-weight states in the generic case with

$$\mathcal{K}_Q(z,\bar{z},L) = \sum_{\substack{\text{highest}\\ \text{weight states}}} z^{\frac{1}{2}(\overline{E}_n+\overline{P}_n)} \bar{z}^{\frac{1}{2}(\overline{E}_n-\overline{P}_n)}. \tag{25}$$

The function $\mathcal{K}_Q$ can also be expressed as

$$\mathcal{K}_Q(z,\bar{z},L) = \mathcal{F}_Q(z,\bar{z},L) - \mathcal{F}_{Q+1}(z,\bar{z},L) \qquad 0 \leqslant Q \leqslant \frac{L}{2}. \tag{26}$$

This relation can be inverted into

$$\mathcal{F}_Q(z,\bar{z},L) = \sum_{j=Q}^{L/2} \mathcal{K}_j(z,\bar{z},L). \tag{27}$$

The partition function for the particular case (8), when q is a root of unity, can be obtained by continuity from the generic case. It is known [1] that, in this case, some representations will mix in higher dimensional representations ('bad' representations) which will contain subrepresentations of zero norm. That could of course lead to problems with physical interpretation. There will, however, still exist representations isomorphic to the usual $SU(2)$ representations with a non-vanishing norm ('good' representations). We can therefore expect that the 'good' sector will lead us to interesting physical models. We therefore need an expression for the partition function $\mathcal{D}_Q(z,\bar{z},L)$ for the highest-weight states from the 'good' sector†. This formula was derived by Pasquier and Saleur (relation (2.9) in [1]) in the context of the open quantum chain. However, their arguments are based purely on group-theoretical grounds and can also be repeated here with the same result. Thus,

$$\begin{aligned}\mathcal{D}_Q(z,\bar{z},L) &= \sum_{r\geqslant 0}(\mathcal{K}_{Q+nr}(z,\bar{z},L) - \mathcal{K}_{n-1-Q+nr}(z,\bar{z},L)) \qquad 0 \leqslant Q < \tfrac{1}{2}(n-1) \\ &= \sum_{r\geqslant 0}\mathcal{K}_{Q+nr}(z,\bar{z},L) - \sum_{r>0}\mathcal{K}_{-Q-1+nr}(z,\bar{z},L)\end{aligned} \tag{28}$$

where from (8) and (20) it follows that $n = m+1$. For later convenience, we transform this formula into another form. We denote the generating function of lowest-weight states by $\mathcal{K}_j$ for $j < 0$. Owing to the symmetries of the Hamiltonian we have

$$\mathcal{K}_j(z,\bar{z},L) = \mathcal{K}_{-j}(z,\bar{z},L). \tag{29}$$

Then (28) can be written as

$$\mathcal{D}_Q(z,\bar{z},L) = \sum_{r\geqslant 0}\mathcal{K}_{Q+nr}(z,\bar{z},L) - \sum_{r<0}\mathcal{K}_{Q+1+nr}(z,\bar{z},L). \tag{30}$$

Analogously to (26), we can express $\mathcal{K}_{-|j|}(z,\bar{z},L)$ as

$$\mathcal{K}_{-|j|}(z,\bar{z},L) = \mathcal{F}_{-|j|}(z,\bar{z},L) - \mathcal{F}_{-|j|-1}(z,\bar{z},L). \tag{31}$$

Using (26) and (31) in (30) we obtain

$$\mathcal{D}_Q(z,\bar{z},L) = \sum_{r\geqslant 0}(\mathcal{F}_{Q+nr}(z,\bar{z},L) - \mathcal{F}_{Q+1+nr}(z,\bar{z},L))$$

† These states can also be characterized (relation (1.19) in [1]) by the condition that they belong to the kernel of S^+ and do not belong to the image of $(S^+)^{n-1}$.

$$-\sum_{r<0}(\mathcal{F}_{Q+1+nr}(z,\bar{z},L)-\mathcal{F}_{Q+nr}(z,\bar{z},L)). \tag{32}$$

It is convenient to introduce the notation

$$\mathcal{G}_Q(z,\bar{z},L)=\sum_{r\in Z}\mathcal{F}_{Q+nr}(z,\bar{z},L). \tag{33}$$

With this notation, the generating function for the 'good' sector can be written as

$$\mathcal{D}_Q(z,\bar{z},L)=\mathcal{G}_Q(z,\bar{z},L)-\mathcal{G}_{Q+1}(z,\bar{z},L). \tag{34}$$

We shall see that $\mathcal{D}_Q(z,\bar{z},L)$ will define the spectrum of a statistical model with non-trivial boundary conditions (sector-dependent). The spectrum of this model is related to the spectrum of our starting quantum chain with the help of (34). The same relation is true in the finite-size scaling limit. In this case, however, we shall be able to determine explicit formulae for the operator content of the resulting model.

3. Quantum chain and the XXZ chain with a toroidal boundary condition

Boundary conditions of the quantum chain are sector-dependent (equations (11) and (12)). One can raise the natural question how the spectrum of the quantum chain is related to the chains with toroidal boundary conditions. As indicated previously, we are particularly interested in the 'good' part of the spectrum of the quantum chain. It turns out that the answer to the above question enables us to use the results of [10, 11] on toroidal models. They provide us with the necessary arguments to show, from relation (34), the results anticipated at the end of the preceding section.

We remind the reader of the results for the toroidal case [10, 11]. The Hamiltonian is defined by

$$H(q,\phi)=-\sum_{i=1}^{L}\left\{\sigma_i^+\sigma_{i+1}^-+\sigma_i^-\sigma_{i+1}^++\frac{q+q^{-1}}{4}(\sigma_i^3\sigma_{i+1}^3)\right\} \tag{35}$$

$$\frac{q+q^{-1}}{2}=\cos\varphi=-\cos\gamma \tag{36}$$

and

$$\sigma_{L+1}^{\pm}=\mathrm{e}^{\mp\mathrm{i}\phi}\sigma_1^{\pm}\qquad \phi\in(-\pi,\pi]. \tag{37}$$

This Hamiltonian commutes with

$$S^z=\sum_{i=1}^{L}\sigma_i^3 \tag{38}$$

and with the translation operator

$$T=\mathrm{e}^{-\mathrm{i}\phi\sigma_1^3/2}P_1P_2\cdots P_{L-1} \tag{39}$$

where P_i, $i=1,\ldots,L-1$ are permutation operators

$$P_i=\sigma_i^+\sigma_{i+1}^-+\sigma_i^-\sigma_{i+1}^++\tfrac{1}{2}(\sigma_i^3\sigma_{i+1}^3+1). \tag{40}$$

The momentum operator is then

$$P=\mathrm{i}\ln T. \tag{41}$$

The BA constraints for this system are [9]

$$Lk_i = 2\pi I_i + \phi - \sum_{\substack{j=1\\ j\neq i}}^{M} \Theta(k_i, k_j) \qquad i = 1, \ldots, M \tag{42}$$

and give

$$E = -\frac{L}{2}\cos\varphi + 2\sum_{i=1}^{M}(\cos\varphi - \cos k_i) \tag{43}$$

$$P = \sum_{i=1}^{M} k_i = \frac{2\pi}{L}\sum_{i=1}^{M} I_i + \frac{M}{L}\phi. \tag{44}$$

We define

$$\phi = 2\pi l \qquad -\tfrac{1}{2} < l \leqslant \tfrac{1}{2}. \tag{45}$$

The finite-size scaling limit of this system is described by the $c = 1$ conformal field theory of the compactified free-boson system with the compactification radius

$$R^2 = 8h \tag{46}$$

where

$$h = \frac{1}{4(1-\gamma/\pi)} \tag{47}$$

and $h \geqslant \frac{1}{4}$.

Let us denote by $E^l_{Q;j}(L)$ and $P^l_{Q;j}(L)$ the eigenvalues of H and P in the sector $S^z = Q$ with a boundary condition defined by $\phi = 2\pi l$. Index $j = 1, \ldots, \binom{L}{Q+L/2}$ enumerates eigenvalues with the same l and Q (some of them can of course coincide). Then, following [10, 11], we can write the expression for the finite-size scaling function of H^l_Q:

$$\begin{aligned}
\mathcal{E}^l_Q(z,\bar{z}) &= \lim_{L\to\infty} \mathcal{E}^l_Q(z,\bar{z},L) \\
&= \lim_{L\to\infty} \sum_{j=1}^{\binom{L}{Q+L/2}} z^{\frac{1}{2}(\overline{E}^l_{Q;j}(L)+\overline{P}^l_{Q;j}(L))}\bar{z}^{\frac{1}{2}(\overline{E}^l_{Q;j}(L)-\overline{P}^l_{Q;j}(L))} \\
&= \sum_{\nu\in Z} z^{[Q+4h(l+\nu)]^2/16h}\bar{z}^{[Q-4h(l+\nu)]^2/16h}\prod_{n=1}^{\infty}(1-z^n)^{-1}(1-\bar{z}^n)^{-1}.
\end{aligned} \tag{48}$$

The symbols $\overline{E}^l_{Q;j}(L)$ and $\overline{P}^l_{Q;j}(L)$ denote the scaled gaps

$$\overline{E}^l_{Q;j}(L) = \frac{L}{2\pi}(E^l_{Q;j}(L) - E^0_{0;0}(L))$$

$$\overline{P}^l_{Q;j}(L) = \frac{L}{2\pi}P^l_{Q;j}(L).$$

It was shown [10, 11] that it was possible to project theories with $c < 1$ by choosing a new ground state with energy $E^{l_0}_{0;j_0}(L)$. The number $j_0 \geqslant 1$ was chosen in such a way that the new ground state gave the contribution $(z\bar{z})^{h(l_0+\nu_0)^2}$ in the partition function (48). The quantity $(l_0 + \nu_0)$ is related to h by the condition

$$c = 1 - \frac{6}{m(m+1)} = 1 - 24h(l_0+\nu_0)^2 \tag{49}$$

where $-\frac{1}{2} < l_0 \leqslant \frac{1}{2}$ and $\nu_0 \in Z$. From (49) it follows that

$$l_0 + \nu_0 = [4hm(m+1)]^{-\frac{1}{2}}. \tag{50}$$

Now new scaled gaps can be defined as

$$\overline{F}^k_{Q;j}(L) = \frac{L}{2\pi}(E^{k(l_0+\nu_0)}_{Q;j}(L) - E^{l_0}_{0;j_0}(L))$$
$$\overline{P}^k_{Q;j}(L) = \frac{L}{2\pi}P^{k(l_0+\nu_0)}_{Q;j}(L).$$

The corresponding finite-size scaling partition function is

$$\begin{aligned}\mathcal{F}^k_Q(z,\bar{z}) &= \lim_{L\to\infty}\mathcal{F}^k_Q(z,\bar{z},L)\\ &= \lim_{L\to\infty}\sum_{j=1}^{\binom{L}{Q+L/2}} z^{\frac{1}{2}(\overline{F}^k_{Q;j}(L)+\overline{P}^k_{Q;j}(L))}\,\bar{z}^{\frac{1}{2}(\overline{F}^k_{Q;j}(L)-\overline{P}^k_{Q;j}(L))}.\end{aligned} \tag{51}$$

The relation (49) gives c as a function of two independent real parameters, h and $l_0 + \nu_0$. According to [10], two classes of $c < 1$ models can be defined imposing the relation

$$l_0 + \nu_0 = \frac{1}{M} - \frac{M}{4h}. \tag{52}$$

They are called R models if $M > 0$ ($R = M$) and L models if $M < 0$ ($L = -M$). From (49) and (52) it follows that

$$\varphi = \frac{\pi m}{R^2(m+1)} \qquad R \text{ models} \tag{53}$$

$$\varphi = \frac{\pi(m+1)}{L^2 m} \qquad L \text{ models.} \tag{54}$$

Our goal is to evaluate (34) extracted from the quantum chain. We thus choose case (53) with $R = 1$ which reproduces our equation (20). In this case,

$$l_0 + \nu_0 = \frac{1}{m+1} \tag{55}$$

and the function $\mathcal{F}^k_Q(z,\bar{z},L)$ has the periodicity properties

$$\mathcal{F}^k_Q(z,\bar{z},L) = \mathcal{F}^{k\pm n}_Q(z,\bar{z},L) \tag{56}$$

where the integer n is given by

$$n = m + 1. \tag{57}$$

Consider the function $\mathcal{G}^k_Q(z,\bar{z},L)$

$$\mathcal{G}^k_Q(z,\bar{z},L) = \sum_{\nu\in Z}\mathcal{F}^k_{Q+\nu n}(z,\bar{z},L) \tag{58}$$

which satisfies

$$\mathcal{G}^k_{Q\pm n}(z,\bar{z},L) = \mathcal{G}^{k\pm n}_Q(z,\bar{z},L) = \mathcal{G}^k_Q(z,\bar{z},L) = \mathcal{G}^{n-k}_{n-Q}(z,\bar{z},L). \tag{59}$$

We define also

$$\mathcal{D}^k_Q(z,\bar{z},L) \equiv \mathcal{G}^k_Q(z,\bar{z},L) - \mathcal{G}^Q_k(z,\bar{z},L) \tag{60}$$

where

$$1 \leqslant k \leqslant m \qquad |Q| \leqslant \min\{k-1, m-k\}. \tag{61}$$

From (48), (49) and (58) one obtains [10,11]

$$\mathcal{D}_Q^k(z,\bar{z}) \equiv \lim_{L\to\infty} \mathcal{D}_Q^k(z,\bar{z},L) = \sum_{r=1}^{m-1} \chi_{r,k-Q}(z)\,\chi_{r,k+Q}(\bar{z}). \tag{62}$$

The symbols $\chi_{r,s}$ denote the character functions of irreducible representations of the Virasoro algebra with highest weights $\Delta_{r,s}$ given by

$$\Delta_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}. \tag{63}$$

The functions $\mathcal{F}_Q^k$ are partition functions of the toroidal chain with the boundary condition

$$\phi = 2\pi\frac{k}{n} = 2\pi\frac{k}{m+1} \qquad \text{in the sector } S^z = Q. \tag{64}$$

On the other hand, we have seen that the quantum chain has boundary conditions given by

$$\phi = 2\varphi(Q+1) = 2\pi\frac{m}{m+1}(Q+1) \quad (\mathrm{mod}\ 2\pi). \tag{65}$$

Comparing (64) with (65), one obtains

$$k = -(Q+1) \quad (\mathrm{mod}\ n) = m - Q \quad (\mathrm{mod}\ n). \tag{66}$$

The highest-weight states for the quantum chain in the sector of charge Q satisfy the same BA equations as the toroidal Hamiltonian with the boundary condition (65). As a consequence, the energy and momenta of the states are simply related (as follows by comparing (9) with (43), and (16) with (44)) by

$$E = E(\text{toroidal}) + \frac{L}{2}\cos\varphi \tag{67}$$

$$P = P(\text{toroidal}) - \varphi\left(Q - 1 + \frac{L}{2}\right). \tag{68}$$

Using (66) and (60) we obtain

$$\mathcal{D}_Q^{m-Q}(z,\bar{z},L) = \mathcal{D}_Q^{-(Q+1)}(z,\bar{z},L) = \mathcal{G}_Q^{-(Q+1)}(z,\bar{z},L) - \mathcal{G}_{-(Q+1)}^{Q}(z,\bar{z},L). \tag{69}$$

However, the relation (34) for the quantum chain, after using the symmetry property

$$\mathcal{G}_Q(z,\bar{z},L) = \mathcal{G}_{-Q}(z,\bar{z},L) \tag{70}$$

has the same form as (69). Indeed, we know from [1] that formula (69) for the toroidal Hamiltonian projects states which are in the kernel of S^+ and not in the image of $(S^+)^{n-1}$. However, that was also the case with relation (34). In fact, the left-hand sides of these two relations are both 'good' highest-weight states with the same charge and the same boundary condition and satisfy the same BA equations. Adding the usual assumption that all 'good' highest-weight states are given by BA states, we conclude that the left-hand sides of (34) and (69) are equal. In other words,

$$\mathcal{D}_Q(z,\bar{z},L) = \mathcal{D}_Q^{m-Q}(z,\bar{z},L). \tag{71}$$

4. Unitary minimal models and the quantum chain

One result of the preceding section is the relation (71) which in combination with (62) leads to

$$\mathcal{D}_Q(z,\bar{z}) = \lim_{L\to\infty} \mathcal{D}_Q(z,\bar{z},L) = \sum_{r=1}^{m-1} \chi_{r,m-2Q}(z)\,\chi_{r,m}(\bar{z}) \tag{72}$$

$$0 \leqslant Q < \frac{m}{2}. \tag{73}$$

The right-hand side of (72) has the form of partition functions of physical systems [16, 17] where $\chi_{r,s}$ are denoted character functions of highest-weight representations of the Virasoro algebra. The quantum parameter φ determines m:

$$\varphi = \frac{\pi m}{m+1}. \tag{74}$$

Accordingly, the construction (72) gives the partition function of a system which consists of a 'good' subset of states of the original quantum chain; this system has the conformal anomaly

$$c = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, \ldots \tag{75}$$

and the operator content can be read from (72) with the help of formula (63). Owing to (75) this system belongs to the unitary series.

We present some simple examples.

4.1. $m = 3$

From (73) it follows that $Q = 0, 1$.

$$\begin{aligned}\mathcal{D}_0(z,\bar{z}) &= \sum_{r=1}^{2} \chi_{r,3}(z)\,\chi_{r,3}(\bar{z}) = \sum_{r=1}^{2}(\Delta_{r,3}, \overline{\Delta}_{r,3}) \\ &= (0,0) + (\tfrac{1}{2}, \tfrac{1}{2})\end{aligned} \tag{76}$$

$$\begin{aligned}\mathcal{D}_1(z,\bar{z}) &= \sum_{r=1}^{2} \chi_{r,1}(z)\,\chi_{r,3}(\bar{z}) = \sum_{r=1}^{2}(\Delta_{r,1}, \overline{\Delta}_{r,3}) \\ &= (\tfrac{1}{2}, 0) + (0, \tfrac{1}{2}).\end{aligned} \tag{77}$$

We have used the usual notation

$$\chi_{r,s}(z)\,\chi_{p,t}(\bar{z}) \equiv (\Delta_{r,s}, \overline{\Delta}_{p,t}). \tag{78}$$

These functions can be identified with the partition functions for given sectors of the Ising chain. In fact, consider the Hamiltonian

$$H = -\tfrac{1}{2} \sum_{j=1}^{L/2} (\sigma_j^3 + \sigma_j^1 \sigma_{j+1}^1) \tag{79}$$

with the boundary conditions

$$\sigma^1_{\frac{L}{2}+1} = (-1)^{\tilde{q}} \sigma^1_1 \qquad \tilde{q} = 0, 1.$$

This Hamiltonian commutes with the operator Σ

$$\Sigma = \sigma_1^3 \cdots \sigma_{L/2}^3. \tag{80}$$

We use $T_q^{\tilde{q}}$ to denote the partition function for the boundary condition defined by $\tilde{q}$ and the eigenvalue $(-1)^q$ of Σ. Their conformal content was obtained in [18]. By comparison,

$$\mathcal{D}_0 = T_0^0 \qquad \mathcal{D}_1 = T_1^1. \tag{81}$$

Thus we see that, for example, T_0^1 and T_1^0 are not contained in this construction. It would be interesting to construct a Hamiltonian containing just these sectors. In fact, this is a non-local Hamiltonian already discussed in [7, 11] which we mention here for completeness:

$$H = -\tfrac{1}{2}\left\{ \sum_{j=1}^{L/2-1} (\sigma_j^3 + \sigma_j^1 \sigma_{j+1}^1) + \sigma_{L/2}^3 + \sigma_{L/2}^1 \sigma_1^1 \Sigma \right\}. \tag{82}$$

4.2. $m = 4$

Again, $Q = 0, 1$.

$$\begin{aligned} \mathcal{D}_0 &= \sum_{r=1}^{3} (\Delta_{r,4}, \overline{\Delta}_{r,4}) \\ &= (0, 0) + (\tfrac{7}{16}, \tfrac{7}{16}) + (\tfrac{3}{2}, \tfrac{3}{2}) \\ \mathcal{D}_1 &= \sum_{r=1}^{3} (\Delta_{r,2}, \overline{\Delta}_{r,4}) \\ &= (\tfrac{3}{5}, 0) + (\tfrac{3}{80}, \tfrac{7}{16}) + (\tfrac{1}{10}, \tfrac{3}{2}). \end{aligned}$$

4.3. $m = 5$

$Q = 0, 1, 2.$

$$\begin{aligned} \mathcal{D}_0 &= \sum_{r=1}^{4} (\Delta_{r,5}, \overline{\Delta}_{r,5}) \\ &= (0, 0) + (\tfrac{2}{5}, \tfrac{2}{5}) + (\tfrac{7}{5}, \tfrac{7}{5}) + (3, 3) \\ \mathcal{D}_1 &= \sum_{r=1}^{4} (\Delta_{r,3}, \overline{\Delta}_{r,5}) \\ &= (\tfrac{1}{15}, \tfrac{2}{5}) + (\tfrac{2}{3}, 0) + (\tfrac{1}{15}, \tfrac{7}{5}) + (\tfrac{2}{3}, 3) \\ \mathcal{D}_2 &= \sum_{r=1}^{4} (\Delta_{r,1}, \overline{\Delta}_{r,5}) \\ &= (3, 0) + (\tfrac{7}{5}, \tfrac{2}{5}) + (\tfrac{2}{5}, \tfrac{7}{5}) + (0, 3). \end{aligned} \tag{83}$$

These are partition functions of the three-state Potts model whose Hamiltonian is given by

$$H = -\frac{2}{3\sqrt{3}} \sum_{j=1}^{N} (\sigma_j + \sigma_j^\dagger + \Gamma_j \Gamma_{j+1}^\dagger + \Gamma_j^\dagger \Gamma_{j+1}) \tag{84}$$

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \omega = \mathrm{e}^{2\pi \mathrm{i}/3}. \tag{85}$$

We introduce the partition functions $T_q^{\tilde{q}}$ corresponding to the boundary condition

$$\Gamma_{N+1} = \omega^{\tilde{q}} \Gamma_1 \qquad q = 0, 1, 2 \tag{86}$$

and the sector q of the charge operator

$$M = \sigma_1 \cdots \sigma_N. \tag{87}$$

Then, comparing (83) with the decomposition (2.37a) of [10], we obtain

$$\mathcal{D}_0 = T^0_{0,+} \qquad \mathcal{D}_1 = T^2_1 = T^1_2 \qquad \mathcal{D}_2 = T^0_{0,-}. \tag{88}$$

The functions $\mathcal{D}_0$ and $\mathcal{D}_2$ have the same boundary condition and the charge q, and the notation $+, -$ distinguishes between two one-dimensional representations of S_3 (see, e.g. [10]). All other partition functions are forbidden in our model. In fact the Hamiltonian for this case can be constructed as

$$H = -\frac{2}{3\sqrt{3}}\left\{\sum_{j=1}^{N-1}(\sigma_j + \sigma_j^\dagger + \Gamma_j\Gamma_{j+1}^\dagger + \Gamma_j^\dagger\Gamma_{j+1}) + \sigma_N + \sigma_N^\dagger + \Gamma_N\Gamma_1^\dagger M + \Gamma_N^\dagger\Gamma_1 M^\dagger\right\} \tag{89}$$

where $N = L/2$. As expected, it is again non-local and implies sector-dependent boundary conditions. We note that by making the replacement $M \to M^\dagger$ in (89), and after an obvious adjustment of multiplicative and additive constants, we obtain the Hamiltonian from [7]. These two Hamiltonians have the same energy spectrum, but momenta of opposite sign and thus a different operator content.

Thus, starting with the closed quantum chain, we have obtained the finite-size scaling limit of the partition functions for definite statistical systems. The corresponding operator content can be read from relation (72) given the deformation parameter q of the original quantum chain.

This is a result we also obtain for finite chains. In this case the explicit character formula is not available. However, we are still in a position to relate the spectra of the two theories through the relation (34):

$$\mathcal{D}_Q(z, \bar{z}, L) = \mathcal{G}_Q(z, \bar{z}, L) - \mathcal{G}_{Q+1}(z, \bar{z}, L). \tag{90}$$

This relation for partition functions implies that the spectrum of our statistical system is contained in the quantum chain and can be obtained from (90). We note that

$$\mathcal{G}_Q(z, \bar{z}, L) = \sum_{r\in Z}\mathcal{F}_{Q+rn}(z, \bar{z}, L) \tag{91}$$

where $\mathcal{F}_Q(z, \bar{z}, L)$ is the partition function of the original system defined by (24).

For illustration, we present energies of the quantum chain with four sites for $m = 3$. The construction (90) and (91) (interpreted here as subtraction between two sets of energy eigenvalues and union of sets, respectively) then gives us energies for the projected statistical system. This system was previously identified as the Ising chain with two sites and with the boundary conditions dependent on the sector and defined in (81). The set of energies of the projected system is a subset of the set of energies of the original system and is underlined in table 1. These energies are indeed also energies of the Ising chain (82), as can be checked numerically. Our numbers are a subset of the numbers in table 3 in [10] for chains with toroidal boundary conditions. We have to expect this since we have shown that the spectrum of the quantum chain is contained in the union of spectra of toroidal Hamiltonians.

In table 2 we have illustrated some features for the $m = 5$ case of the three-state Potts model with two sites and sector-dependent boundary conditions. We remark that in both cases the allowed boundary conditions are those permitted by the symmetry on duality transformations [19]:

$$H_q^{\tilde{q}} = H_{\tilde{q}}^q$$

We shall consider this point elsewhere.

Table 1. Scaled energy gaps defined by (22) for the quantum chain with four sites, $m = 3$. The levels which are underlined correspond to the Ising model (82) with two sites.

$\mathcal{G}_0$	$\mathcal{G}_1$	$\mathcal{G}_2$
$\underline{0.000\,000}$	—	—
0.450 158	$\underline{0.450\,158}$	—
0.450 158	$\underline{0.450\,158}$	—
$\underline{0.900316}$	—	—
1.086 778	1.086 778	1.086 778
1.086 778	1.086 778	1.086 778

Table 2. Scaled energy gaps defined by (22) for the quantum chain with four sites, $m = 5$. The levels which are underlined correspond to the three-state Potts model (89) with two sites.

$\mathcal{G}_0$	$\mathcal{G}_1$	$\mathcal{G}_2$
$\underline{0.000\,000}$	—	—
0.424 413	$\underline{0.424\,413}$	—
0.579 759	$\underline{0.579\,759}$	—
$\underline{0.848\,826}$	—	—
1.004 172	$\underline{1.004\,172}$	—
1.159 518	1.159 518	$\underline{1.159\,518}$

Another interesting question concerns the properties of the model defined with (72) under the modular group. Of course, due to the boundary properties of our model, we do not expect invariance on the full modular group but eventually on a subgroup (compare [20, 21]). Indeed, we can show that all functions D_Q are invariant on the modular transformation T^{m+1} ($T^{(m+1)/2}$) for m even (odd) where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This can be obtained by straightforward application of the property [17]

$$T[\chi_{r,s}] = \mathrm{e}^{2\pi\mathrm{i}(\Delta_{r,s}-c/24)}\chi_{r,s}$$

on the relation (72). We have checked for $m = 3, 4, 5$ that all D_Q are also invariant on U^{m+1} where

$$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

T^{m+1} and U^{m+1} generate the subgroup

$$\Gamma(m+1) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad (\mathrm{mod}\ m+1).$$

Let us elaborate further on a particular example $m = 5$ (three-state Potts). In this case (m is odd) we have invariance on U^6 and T^3. Particular combinations of D_Q functions can have higher invariances. So, for example, $D_0 + D_2$ and D_1 are invariant on U^3 and that means on $\Gamma(3)$. Thus, for example, the partition function Z for the non-local Potts Hamiltonian (89) is given by

$$Z = D_0 + 2D_1 + D_2$$

so it is invariant on $\Gamma(3)$.

5. Conclusion

We have treated the closed quantum invariant chain for the quantum parameter $q = e^{i\varphi}$, $\varphi = \pi m/(m+1)$, $m = 3, 4, \ldots$. This model has the conformal anomaly [7]

$$c = 1 - \frac{6}{m(m+1)}.$$

The Hamiltonian also has the property that it implies sector-dependent boundary conditions. We have shown that from the partition function of this theory we can construct partition functions of well defined statistical systems. In particular, the spectra of these are subsets of the spectrum of the quantum chain and can be obtained using (34). These formulae have been obtained using the theory of representations of quantum groups, keeping the 'good' states and omitting the 'bad' states.

We have shown how our construction is related to the well known projection mechanism of statistical models from Hamiltonians with toroidal boundary conditions.

Finally, using this relation we have been able to obtain partition functions in the finite-size scaling limit. This has enabled us to find the operator content of the systems constructed from the quantum chain. These systems belong to the family of unitary minimal models. These properties have been illustrated in a few particular cases ($m = 3, 4, 5$).

Acknowledgments

One of us (SP) would like to thank V Rittenberg for permanent interest and H Grosse and P Martin for precious discussions.

References

[1] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[2] Batchelor M T, Mezincescu L, Nepomechie R I and Rittenberg V 1990 *J. Phys. A: Math. Gen.* **23** L141
[3] Meljanac M, Mileković M and Pallua S 1991 *J. Phys. A: Math. Gen.* **24** 581
[4] Batchelor M T and Kuniba A 1991 *J. Phys. A: Math. Gen.* **24** 2599
[5] Martin P and Rittenberg V 1992 *Int. J. Nucl. Phys.* A **7** Suppl 1B, 797
[6] Martin P P 1991 *Potts Models and Related Problems in Statistical Mechanics* (Singapore: World Scientific)
[7] Grosse H, Pallua S, Prester P and Raschhofer E 1994 *J. Phys. A: Math. Gen.* **27** 4761
[8] Karowski M and Zapletal A 1994 *Nucl. Phys.* B **419** [FS] 567
[9] Alcaraz F C, Barber M N and Batchelor M N 1988 *Ann. Phys.* **182** 280
[10] Alcaraz F C, Grimm U and Rittenberg V 1989 *Nucl. Phys.* B **316** 735
[11] Grimm U and Schütz G 1993 *J. Stat. Phys.* **71** 923
[12] Bazhanov V V and Reshetikhin N Yu 1989 *Int. J. Mod. Phys.* A **4** 115
[13] Schütz G 1993 *J. Phys. A: Math. Gen.* **26** 4555
[14] Jüttner G and Karowski M 1994 *Nucl. Phys.* B **430** [FS] 615
[15] Reshetikhin N and Smirnov F 1990 *Commun. Math. Phys.* **131** 157
[16] Rocha-Caridi A 1985 *Vertex Operators in Mathematics and Physics* ed J Lepowski, S Mandelstam and I Singer (Berlin: Springer)
[17] Capelli A, Itzykson C and Zuber J B 1987 *Nucl. Phys.* B **280** [FS18] 445
[18] von Gehlen G and Rittenberg V 1986 *J. Phys. A: Math. Gen.* **19** L625
[19] von Gehlen G, Rittenberg V and Ruegg H 1985 *J. Phys. A: Math. Gen.* **19** 107
[20] Zuber J-B 1986 *Phys. Lett.* **176B** 127
[21] Cardy J L 1986 *Nucl. Phys.* B **275** [FS17] 200